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A New Competitive Ratio of the Harmonic Algorithm for a k-Server Problem with Parallel Requests and Unit Distances

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1 Introduction

In this paper a generalized k-server problem with parallel requests where several servers can also be located on one point is discussed. The investigation of the generalized k-server problem was initiated by an operations research problem which consists of optimal conversions of machines or moulds (see [4] or [8]). It is sensible in the case of parallel requests to distinguish the surplus-situation where the request can be completely fulfilled by means of the k servers and the scarcity-situation where the request cannot be completely met.

The k-server problem was introduced by Manasse, McGeoch and Sleator [11]. Meanwhile it is the most studied problem in the area of competitive online problems. Historical notes on k-server problems can be found in the book by A. Borodin and R. El-Yaniv [3] (sections 10.9 and 11.7) or also in the paper by Y. Bartal and E. Grove [2]. There the two important results are the competitiveness of the deterministic work-function algorithm (see E. Koutsoupias and C. Papadimitriou [9]) and of the randomized Harmonic k-server algorithm against an adaptive online adversary (see Y. Bartal and E. Grove [2]).

The work-function algorithm is an inefficient algorithm (with a good competitive ratio). In contrast the Harmonic k-server algorithm is memoryless and time-efficient. For this reason we first want to focus on a corresponding Harmonic k-server algorithm for the generalized k-server problem.

If one tries to generalize the proof by Y. Bartal and E. Grove [2] several subchains with different length must be considered and one will see that the computation of the weights $f(j)$ is not possible. In this paper we consider the general k-server problem in the case of unit distances. Using rough estimations of numbers of certain partitions we have shown in [7] that a corresponding Harmonic algorithm is competitive. The (usual) k-server problem with unit distances is known as the paging problem and the Harmonic k-server algorithm as RAND algorithm (see [3], chapters 3 and 4). Raghavan and Snir have shown that the RAND algorithm is k competitive against an adaptive online adversary. Although there can occur a

lot more feasible requests in the case of the generalized k-server problem we will show in this paper (using detailed considerations related to sets of certain partitions) that the corresponding Harmonic k-server algorithm is $\max\{k, R(k) - k + 1\}$ competitive (where $R(k)$ is a bound of the requests related to the scarcity-situation, see Theorem 1) and k competitive (just as RAND), if only the surplus-situation is allowed.

2 The formulation of the model

¹ Let $k \geq 1$ be an integer, and $M = (M, d)$ be a finite metric space where M is a set of points with $|M| = N$. An algorithm controls k mobile servers, which are located on points of M . Several servers can be located on one point. The algorithm is presented with a sequence $\sigma = r^1, r^2, \dots, r^n$ of requests where a request r is defined as an N -ary vector of integers with $r_i \in \{0, 1, \dots, k\}$ ("parallel requests"). The request means that r_i server are needed on point i ($i = 1, 2, \dots, N$). We say a request r is served if $\left\{ \begin{array}{l} \text{at least} \\ \text{at most} \end{array} \right\} r_i$ servers lie on i ($i = 1, 2, \dots, N$) in case $\left\{ \begin{array}{l} C[r, k] \\ C[k, r] \end{array} \right\}$. $C[r, k]$

denotes the case $\sum_{i=1}^N r_i \leq k$ (surplus-situation, the request can be completely

fulfilled) and $C[k, r]$ denotes the case $\sum_{i=1}^N r_i \geq k$ (scarcity-situation, the request cannot be completely met, however it should be met as much as possible). By moving servers, the algorithm must serve the requests r^1, r^2, \dots, r^n sequentially. For any request sequence σ and any generalized k-server algorithm $ALG_{p(allel)}$, $ALG_p(\sigma)$ is defined as the total distance (measured by the metric d) moved by the ALG_p 's servers in servicing σ .

In this paper we will show that the corresponding Harmonic k-server algorithm attains a competitive ratio of $\max\{k, R(k) - k + 1\}$ (see Theorem 1) against an adaptive online adversary in the case of unit distances (for the definitions of competitive ratio and adaptive online adversary see [2] or [3], sections 4.1 and 7.1). Analogous to [3], p. 152 working with lazy algorithms ALG_p is sufficient. For that reason we define the set of feasible servers positions with respect to s and r in the following way

$$\begin{aligned} & \hat{A}_{N;k}(s, r) \\ &= \left\{ s' \in P_N(k) \left| \begin{array}{l} r_i \leq s'_i \leq \max\{s_i, r_i\}, \ i = 1, \dots, N, \text{ in } C[r, k] \\ \min\{s_i, r_i\} \leq s'_i \leq r_i, \ i = 1, \dots, N, \text{ in } C[k, r] \end{array} \right. \right\} \end{aligned} \quad (1)$$

¹For basic knowledge of (usual) k-server problems see also [3], chapters 10 and 11 for example.

$$\text{where } P_N(k) := \left\{ s \in \mathbb{Z}_+^n \mid \sum_{i=1}^N s_i = k \right\}. \quad (2)$$

The metric d implies that $(P_N(k), \hat{d})$ is also a finite metric space where \hat{d} are the optimal values of the classical transportation problems with availabilities s and requirements s' of $P_N(k)$: $\sum_{i=1}^N \sum_{j=N}^N d(i, j) x_{ij} \rightarrow \min$

subject to $\sum_{j=1}^N x_{ij} = s_i \forall i, \sum_{i=1}^N x_{ij} = s'_j \forall j, x \in \mathbb{Z}_+^n \times \mathbb{Z}_+^n$ (see [6], Lemma 3.6).

The corresponding $HARMONIC_p$ k -server algorithm operates as follows: Serve a (not completely covered) request r with randomly chosen servers so that for the (new) server positions $s' \in \hat{A}_{N;k}(s, r)$ is valid with respect to the previous server positions s and the request r . More precisely, $HARMONIC_p$ leads to $s' \in \hat{A}_{N;k}(s, r)$ with probability

$$\frac{\frac{1}{\hat{d}(s, s')}}{\sum_{s'' : s'' \in \hat{A}_{N;k}(s, r)} \frac{1}{\hat{d}(s, s'')}}. \quad (3)$$

3 The competitiveness of $HARMONIC_p$ in case of unit distances

Unit distances means that $d(i, j) = 1 \forall i \neq j$. Thus, $\hat{d}(s, s') = \sum_{i=1}^N \frac{1}{2} |s_i - s'_i|$ for

$$s, s' \in P_N(k) \text{ follows and (1) yields } \hat{d}(s, s') = \begin{cases} \sum_{i: r_i^t > s_i} (r_i^t - s_i) & \text{in } C[r, k] \\ \sum_{i: r_i^t < s_i} (s_i - r_i^t) & \text{in } C[k, r] \end{cases}$$

for every $s' \in \hat{A}_{N;k}(s, r)$. Then $s' \in \hat{A}_{N;k}(s, r)$ is chosen randomly and uniformly with probability $\frac{1}{|\hat{A}_{N;k}(s, r)|}$ among all elements of $\hat{A}_{N;k}(s, r)$ by $HARMONIC_p$.

In [7] can be found an example which shows that in order to prove the competitiveness an additional assumption (as $\sum_{i \in M} r_i^t \leq R(k)$ in the following theorem) in the case $C[k, r^t]$ is necessary.

Theorem 1. *The $HARMONIC_p$ k -server algorithm attains a competitive ratio of $C(k) = \max \{k, R(k) - k + 1\}$ against an adaptive online adversary in case of unit distances if $\sum_{i \in M} r_i^t \leq R(k) \forall t$ for given $R(k) > k$.*

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²This condition is important for case $C[k, r^t]$. (According to the introduced model $\sum_{i \in M} r_i^t \leq k$ is true in case $C[r^t, k]$.) See also the above mentioned example.

Proof. We will use a potential function (see [2]) to prove the statement. In case of unit distances it is sufficient to use the following simple potential function

$$\Phi(s, s') := \hat{f} \sum_{i=1}^N \frac{1}{2} |s_i - s'_i| (= \hat{f} \hat{d}(s, s')), \quad s, s' \in P_N(k). \quad (4)$$

At the beginning let $\hat{f} \geq 0$. We will solve for \hat{f} later. More precisely and analogous to Bartal and Grove, let Φ_t denote the value of Φ at the end of the t -th step (corresponding to the t -th request r^t in the request sequence) and let Φ_t^\sim denote the value of Φ after the first stage of the t -th step (i.e., after the adversary's move and before the algorithm's move). In cases $C[r^t, k]$ and $C[k, r^t]$ we will show the following properties (see [2], pages 4 and 5)

$$\Phi \geq 0 \quad (5)$$

$$\Phi_t^\sim - \Phi_{t-1} \leq C(k)D_t, \quad (6)$$

where D_t denotes the distance moved by the offline servers (controlled by the adversary) to serve the request in the t -th step.

$$E(\Phi_t^\sim - \Phi_t) \geq E(Z_t), \quad (7)$$

where Z_t represents the cost which incurred by the online algorithm to serve the request in the t -th step.

If we can show that the potential function satisfies these properties then $HARMONIC_p$ is $C(k)$ competitive.

In the following let

- $\bar{s} (\in P_N(k))$ denote the (offline) servers position controlled by the adversary at the end of the $(t-1)$ -th step (i.e., at the beginning of the t -th step)
- $s (\in P_N(k))$ denote the (online) servers position controlled by the algorithm at the beginning of the t -th step
- $s' (\in \hat{A}_{N;k}(s, r^t))$ denote the (online) servers position at the end of the t -th step and
- $\bar{s}' (\in P_N(k))$ denote the (offline) servers position controlled by the adversary after the first stage of the t -th step.

Proof of (5) and (6):

(5) is straightforward if $\hat{f} \geq 0$. (6) follows by means of the triangle-equation of the metric \hat{d} :

$$\hat{f}\hat{d}(s, \bar{s}') - \hat{f}\hat{d}(s, \bar{s}) \leq \hat{f}\hat{d}(\bar{s}, \bar{s}') = \hat{f}D_t \leq C(k)D_t \text{ if } C(k) \geq \hat{f}.$$

Proof of (7) in case $C[r^t, k]$:

In this case we will show that $\hat{f} = k$ (and hence each $\hat{f} \geq k$) satisfies the property (7).

For unit distances it follows that

$$\Phi_t^\sim(s, \bar{s}') = \hat{f} \sum_{i: \bar{s}'_i > s_i} (\bar{s}'_i - s_i) = \hat{f} \sum_{i: \bar{s}'_i < s_i} (s_i - \bar{s}'_i) \quad \forall s' \in \hat{A}_{N;k}(s, r^t) \quad (8)$$

as well as

$$\begin{aligned} Z_t(s, s') &= \sum_{i: r_i^t > s_i} (r_i^t - s_i) \quad \forall s' \in \hat{A}_{N;k}(s, r^t) \\ &=: Z_t(s, r^t) \end{aligned} \quad (9)$$

and (7) is equivalent to

$$\Phi_t^\sim - E(\Phi_t) \geq Z_t. \quad (7a)$$

We will reduce the problem in several steps and consider, firstly, certain cases for which the proof is simple. Finally, a remaining reduced problem will be investigated using properties of certain partitions of integers.

Next the set $M = \{i = 1, \dots, N\}$ of points is partitioned in relation to s, \bar{s}'_i, r^t, s'_i in case $C[r^t, k]$ where $r_i^t \leq s'_i \leq \max\{r_i^t, s_i\}$ for $i = 1, \dots, N$:

$$\begin{aligned} M_I &= \{i \in M \mid s_i > s'_i \geq \bar{s}'_i = r_i^t \text{ or } s_i \geq s'_i > \bar{s}'_i = r_i^t\}, \\ M_{IIa} &= \{i \in M \mid s_i > s'_i \geq \bar{s}'_i > r_i^t \text{ or } s_i \geq s'_i > \bar{s}'_i > r_i^t\}, \\ M_{IIb} &= \{i \in M \mid s_i > \bar{s}'_i > s'_i \geq r_i^t\}, \\ M_{III} &= \{i \in M \mid r_i^t \leq s'_i < s_i \leq \bar{s}'_i \text{ or } r_i^t < s'_i = s_i \leq \bar{s}'_i\}, \\ M_{IV} &= \{i \in M \mid s_i \leq r_i^t = s'_i \leq \bar{s}'_i\} = \{i \in M \mid s_i \leq r_i^t\}. \end{aligned}$$

A first reduced model (reduction in 3 steps):

The quantities of the property (7a) will not change by the following manipulations. Particularly, the essential structure of $\hat{A}_{N;k}$ (see (1)) and $|\hat{A}_{N;k}|$ will also not change. k must be reduced in corresponding way.

1. $\Delta_i := \min\{s_i, r_i^t, \bar{s}'_i\}$ for $i \in M$, $k := k - \sum_{i \in M} \Delta_i$,
 $s_i := s_i - \Delta_i, \bar{s}'_i := \bar{s}'_i - \Delta_i, s'_i := s'_i - \Delta_i, r_i^t := r_i^t - \Delta_i$ for $i \in M$
2. $\bar{s} := \sum_{i \in M_{III}} (\bar{s}'_i - s_i) + \sum_{i \in M_{IV}} (\bar{s}'_i - r_i^t)$,
 $\bar{s}'_i := s_i$ for $i \in M_{III}$, $\bar{s}'_i := r_i^t (= s'_i)$ for $i \in M_{IV}$. Temporarily,
we set $s_{i'} = s'_{i'} = r_{i'}^t := 0, \bar{s}'_{i'} := \bar{s}$ for an additional $i' \in \{i'\} =: M_V$.
3. We replace the elements of M_{IV} by one element i where
 $r^* := \sum_{i \in M_{IV}} r_i^t =: s'_i =: \bar{s}'_i$.

Then with regard to s, s', \bar{s}', r^t the following possibilities remain for the reduced model:

$$s_i > s'_i \geq \bar{s}'_i = 0 = r_i^t \text{ or } s_i \geq s'_i > \bar{s}'_i = 0 = r_i^t \text{ for } i \in M_I,$$

$$\begin{aligned}
s_i &> s'_i \geq \bar{s}'_i > 0 = r_i^t \text{ or } s_i \geq s'_i > \bar{s}'_i > 0 = r_i^t \text{ for } i \in M_{IIa}, \\
s_i &> \bar{s}'_i > s'_i \geq r_i^t = 0 \text{ for } i \in M_{IIb}, \\
r_i^t &= 0 \leq s'_i < s_i = \bar{s}'_i \text{ or } r_i^t = 0 < s'_i = s_i = \bar{s}'_i \text{ for } i \in M_{III}, \\
s_i &= 0 < r^* = s'_i = \bar{s}'_i \text{ for } i \in M_{IV} = \{i\}, \\
s_{i'} &= s'_{i'} = r_{i'}^t = 0, \bar{s}'_{i'} = \bar{s} \text{ for } i' \in M_V = \{i'\}
\end{aligned}$$

where

$$r^* = \sum_{i \in M_I \cup M_{IIa} \cup M_{IIb} \cup M_{III}} (s_i - s'_i). \quad (10)$$

Furthermore,

$$Z_t(s, s') = r^* \forall s' \in \hat{A}_{N;k}(s, r) \text{ (see (9))},$$

$$\Phi_t^\sim(s, \bar{s}') = \hat{f}(r^* + \bar{s}) \text{ (see (8))} \quad (11)$$

and

$$\begin{aligned}
(0 \leq) \Phi_t(s', \bar{s}') &= \hat{f} \left(\sum_{i \in M_{IIb} \cup M_{III}} (\bar{s}'_i - s'_i) + \bar{s} \right) \\
&= \hat{f} \left(\sum_{i \in M_{IIb}} (s_i - s'_i) - \sum_{i \in M_{IIb}} (s_i - \bar{s}'_i) + \sum_{i \in M_{III}} (s_i - s'_i) + \bar{s} \right) \\
&= \hat{f} \left(r^* - \sum_{i \in M_I \cup M_{IIa}} (s_i - s'_i) - \sum_{i \in M_{IIb}} (s_i - \bar{s}'_i) + \bar{s} \right) (\leq \hat{f}(r^* + \bar{s}))
\end{aligned} \quad (12)$$

follow (the last equation by means of (10)).

$\hat{f}\bar{s}$ vanishes in the difference $\Phi_t^\sim(s, \bar{s}') - \Phi_t(s', \bar{s}')$. That's why we consider an unbalanced reduced model without M_V and with the difference of \bar{s} between $\sum_{i \in M} s_i$ and $\sum_{i \in M} \bar{s}'_i$. $Z_t(s, s') = r^* \forall s' \in \hat{A}_{N;k}(s, r)$ remains valid. We set $\Phi_t^\sim(s, \bar{s}') = \hat{f} r^*$ and $\Phi_t(s', \bar{s}') = \hat{f} \hat{d}'(s', \bar{s}')$ where

$$\hat{d}'(s', \bar{s}') = \sum_{i \in M_{IIb} \cup M_{III}} (\bar{s}'_i - s'_i) \text{ for } s' \in \hat{A}_{N;k}(s, r). \quad (13)$$

In this way $\Phi_t^\sim(s, \bar{s}') - \Phi_t(s', \bar{s}')$ does not change.

Clearly that

$$k = \sum_{i \in M_I \cup M_{IIa} \cup M_{IIb} \cup M_{III}} s_i \geq r^* + \sum_{i \in M_{IIa} \cup M_{IIb} \cup M_{III}} \bar{s}'_i \quad (14)$$

is right for the unbalanced model.

We can then conclude from (12) that

$$\Phi_t(s', \bar{s}') = r^* \Leftrightarrow s_i - s'_i = 0 \text{ for } i \in M_I \cup M_{IIa}, M_{IIb} = \emptyset, \quad (15)$$

thus

$$\Phi_t(s', \bar{s}') = r^* \Leftrightarrow r^* = \sum_{i \in M_{III}} (s_i - s'_i) \text{ (see (10))} \Rightarrow \sum_{i \in M_{III}} s_i \geq r^*. \quad (16)$$

With regard to the reduced model (7a) is equivalent to

$$\hat{f} r^* - \hat{f} \sum_{s' \in \hat{A}_{N;k}(s,r)} E (\hat{d}'(s', \bar{s}')) \geq r^* \text{ and hence also to } \frac{r^*}{r^* - \sum_{s' \in \hat{A}_{N;k}(s,r)} E (\hat{d}'(s', \bar{s}'))} \leq \hat{f}.$$

Finally the conjecture $\hat{f} = k = C(k)$ ³ in case $C[r^t, k]$ leads to the following representation of (7a):

$$\sum_{s' \in \hat{A}_{N;k}(s,r)} E (\hat{d}'(s', \bar{s}')) \leq r^* \frac{k-1}{k}. \quad (17)$$

The inequality

$$\sum_{s' \in \hat{A}_{N;k}(s,r)} E (\hat{d}'(s', \bar{s}')) \leq r^* - 1 \quad (18)$$

is sufficient for the validity of (17) since $k \geq r^*$, see (14).

Case: $\exists s' \in \hat{A}_{N;k}(s, r) : M_{IIb} \neq \emptyset$

Let $\hat{A}_{N;k}^I(s, r) = \left\{ s' \in \hat{A}_{N;k}(s, r) \mid M_{IIb} = \emptyset \right\}$ and

$$\hat{A}_{N;k}^{II}(s, r) = \hat{A}_{N;k}(s, r) \setminus \hat{A}_{N;k}^I(s, r).$$

Then (15) implies that $\sum_{s' \in \hat{A}_{N;k}^{II}(s,r)} E (\hat{d}'(s', \bar{s}')) \leq r^* - 1$.

Using the relation

$$\sum_{s' \in \hat{A}_{N;k}(s,r)} E (\hat{d}'(s', \bar{s}')) = \sum_{s' \in \hat{A}_{N;k}^I(s,r)} E (\hat{d}'(s', \bar{s}')) \frac{|\hat{A}_{N;k}^I(s,r)|}{|\hat{A}_{N;k}(s,r)|} + \sum_{s' \in \hat{A}_{N;k}^{II}(s,r)} E (\hat{d}'(s', \bar{s}')) \frac{|\hat{A}_{N;k}^{II}(s,r)|}{|\hat{A}_{N;k}(s,r)|}$$

we get

$$a) \quad \sum_{s' \in \hat{A}_{N;k}(s,r)} E (\hat{d}'(s', \bar{s}')) \leq r^* - 1 \text{ if } \sum_{s' \in \hat{A}_{N;k}^I(s,r)} E (\hat{d}'(s', \bar{s}')) \leq r^* - 1 \text{ and}$$

the conjecture (17) is true according to (18),

$$b) \quad \sum_{s' \in \hat{A}_{N;k}(s,r)} E (\hat{d}'(s', \bar{s}')) < \sum_{s' \in \hat{A}_{N;k}^I(s,r)} E (\hat{d}'(s', \bar{s}')) \text{ if}$$

$$\sum_{s' \in \hat{A}_{N;k}^I(s,r)} E (\hat{d}'(s', \bar{s}')) > r^* - 1 \text{ and it is sufficient to consider a reduced}$$

model with $M_{IIb} = \emptyset \forall s' \in \hat{A}_{N;k}(s, r)$.

Case: $M_{IIb} = \emptyset \forall s' \in \hat{A}_{N;k}(s, r)$

In this case we can use a further reduction step:

$$s_i := s_i - \bar{s}'_i, \quad s'_i := s'_i - \bar{s}'_i, \quad \bar{s}'_i := 0 \text{ for } i \in M_{IIa}, \quad k := \sum_{i \in M_I \cup M_{IIa} \cup M_{III}} s_i.$$

Then the set M_{IIa} can be integrated into the set M_I and a reduced model with the following possibilities in relation to s, s', \bar{s}', r^t remains:

$$s_i > s'_i \geq \bar{s}'_i = 0 = r_i^t \text{ or } s_i \geq s'_i > \bar{s}'_i = 0 = r_i^t \text{ for } i \in M_I,$$

³It is unproblematic to use the reduced k since the original k is greater or equal than the reduced.

$$\begin{aligned} r_i^t = 0 \leq s'_i < s_i = \bar{s}'_i \text{ or } r_i^t = 0 < s'_i = s_i = \bar{s}'_i \text{ for } i \in M_{III}, \\ s_i = 0 < r^* = s'_i = \bar{s}'_i \text{ for } i \in M_{IV}. \end{aligned}$$

The union of sets in the formulas (14) and (10) must then be replaced by $M_I \cup M_{III}$ or M_{III} . Furthermore (10) leads to

$$k \geq r^* \quad (19)$$

and

$$\sum_{i \in M_I} s_i \geq r^*. \quad (20)$$

If $\sum_{i \in M_{III}} s_i \leq r^* - 1$ then $E_{s' \in \hat{A}_{N;k}(s,r)}(\hat{d}'(s', \bar{s}')) \leq r^* - 1$ follows from (16) and the conjecture (17) is true according to (18). Hence it remains to investigate the reduced problem with $M_{III} = \emptyset$ where (20) and

$$\sum_{i \in M_{III}} s_i \geq r^* \quad (21)$$

are assumed.

Let us set: $M_I = \{1, \dots, m\}$, $M_{III} = \{m+1, \dots, m+n\}$, $M_{IV} = \{m+n+1\}$ and $s^I = (s_{m+1}, \dots, s_{m+n})$, $s^{II} = (s_1, \dots, s_m)$.

In order to prove the conjecture for the reduced problem we use ordered restricted partitions of integers x (compositions) into exactly n non-negative parts (written as vectors):

$$P_s^n(x) := \left\{ \bar{x} \in \mathbb{Z}_+^n \mid 0 \leq \bar{x}_i \leq s_i \text{ for } i = 1, \dots, n, \sum_{i=1}^n \bar{x}_i = x \right\}, \quad p_s^n(x) := |P_s^n(x)| \text{ for } x \in \mathbb{Z}_+ \text{ and given } n \in \mathbb{N}, s \in \mathbb{Z}_+^n.$$

A one-to-one correspondence between the elements of the sets $\hat{A}_{N;k}(s, r)$ and $P_{(s^{II}, s^I)}^{n+m}(r^*)$ is defined by:

$$\{s' \leftrightarrow \bar{x}\} \Leftrightarrow \{s_i - s'_i = \bar{x}_i \text{ for } i = 1, \dots, n+m\}. \quad (22)$$

$$|\hat{A}_{N;k}(s, r)| = p_{(s^{II}, s^I)}^{n+m}(r^*) = \sum_{x=0}^{r^*} p_{s^I}^n(x) p_{s^{II}}^m(r^* - x) \text{ (pay attention to}$$

$$(21) \text{ and } (20)) \text{ and } \hat{d}'(s', \bar{s}') = \sum_{i \in M_{III}} (\bar{s}'_i - s'_i) = \sum_{i=m+1}^{m+n} (s_i - s'_i) = \sum_{i=m+1}^{m+n} \bar{x}_i$$

(see (13)) imply the following representation of the conjecture (17):

$$E_{s' \in \hat{A}_{N;k}(s,r)}(\hat{d}'(s', \bar{s}')) = \frac{\sum_{x=0}^{r^*} x p_{s^I}^n(x) p_{s^{II}}^m(r^* - x)}{\sum_{x=0}^{r^*} p_{s^I}^n(x) p_{s^{II}}^m(r^* - x)} \leq r^* \frac{k-1}{k}, \quad (23)$$

where (20) and (21) are assumed.

In order to prove (23) we use the following properties of $p_s^n(x)$ and the following relationship (29):

- $$p_s^n(x) = p_s^n\left(\sum_{i=1}^n s_i - x\right) \text{ (symmetry)}$$
 (24)

- $p_s^n(x)$ is monotonically increasing for
$$x \in \{0, 1, \dots, \max\{s_n, \lceil \frac{1}{2} \sum_{i=1}^n s_i \rceil\}\}$$
 (25)

- $p_s^n(x)$ is strictly increasing for $x \in \{0, 1, \dots, \min\{\sum_{i=1}^{n-1} s_i; \lfloor \frac{1}{2} \sum_{i=1}^n s_i \rfloor\}\}$ (26)

- $$(r^*+1)p_s^n(r^*+1) \leq n \sum_{x=0}^{r^*} p_s^n(x), r^* \in \mathbb{Z}_+$$
 (27)

- $$\frac{(x+1)p_s^n(x+1)+yp_s^n(y)}{p_s^n(x+1)+p_s^n(y)} \leq \frac{(y+1)p_s^n(y+1)+xp_s^n(x)}{p_s^n(y+1)+p_s^n(x)} \leq y+2$$
 for $x < y$ and $y+1, y+2 \leq \max\{s_n, \lceil \frac{1}{2} \sum_{i=1}^n s_i \rceil\}$ (28)

•

Let $a_i > 0$ for $i = 0, \dots, l$, $b_0 \geq b_1 \geq \dots \geq b_l > 0$ and $A_0 \leq A_1 \leq \dots \leq A_l$. Then $\frac{A_0 a_0 b_0 + \dots + A_l a_l b_l}{a_0 b_0 + \dots + a_l b_l} \leq \frac{A_0 a_0 + \dots + A_l a_l}{a_0 + \dots + a_l}$ is valid. (29)

(See the following Remarks 1 for (24), \dots , (27) and (29). Simple computations yield (28).)

Case $p_{s_I}^n(r^* - 1) > p_{s_I}^n(r^*)$:

Together with (24) and (26) the relations $r^* > 1$ and $p_{s_I}^n(r^* - 2) \geq p_{s_I}^n(r^*)$ follow. The last inequality implies $p_{s_I}^n(r^* - 2) p_{s_{II}}^m(2) \geq p_{s_I}^n(r^*) = p_{s_I}^n(r^*) p_{s_{II}}^m(0)$ and $\frac{(r^*-2) p_{s_I}^n(r^*-2) p_{s_{II}}^m(2) + r^* p_{s_I}^n(r^*) p_{s_{II}}^m(0)}{p_{s_I}^n(r^*-2) p_{s_{II}}^m(2) + p_{s_I}^n(r^*) p_{s_{II}}^m(0)} \leq r^* - 1$.

Hence
$$E_{s' \in \hat{A}_{N;k}(s,r)}(\hat{d}'(s', \bar{s}')) = \frac{\sum_{x=0}^{r^*} x p_{s_I}^n(x) p_{s_{II}}^m(r^* - x)}{\sum_{x=0}^{r^*} p_{s_I}^n(x) p_{s_{II}}^m(r^* - x)} \leq r^* - 1$$
, and (23) is valid

in this case (see (18)).

Case $p_{s_I}^n(r^* - 1) \leq p_{s_I}^n(r^*)$:

We show:

I

$$\frac{\sum_{x=0}^{r^*} x p_{s_I}^n(x) p_{s_{II}}^m(r^* - x)}{\sum_{x=0}^{r^*} p_{s_I}^n(x) p_{s_{II}}^m(r^* - x)} \leq \frac{\sum_{x=0}^{r^*} x p_{s_I}^n(x)}{\sum_{x=0}^{r^*} p_{s_I}^n(x)} \quad (30)$$

(this means $|M_I| = m = 1$ yields an upper bound)

II

$$\frac{\sum_{x=0}^{r^*} x p_{sI}^n(x)}{\sum_{x=0}^{r^*} p_{sI}^n(x)} \leq r^* \frac{k-1}{k}. \quad (31)$$

To I: Let us set $l = \min\{r^*, \left\lfloor \frac{1}{2} \sum_{i=1}^m s_i^{II} \right\rfloor\}$, $L = \max\{0, 2r^* - \sum_{i=1}^m s_i^{II}\}$,

$$\bar{L} = \left\lfloor \frac{L}{2} \right\rfloor = \max\{0, r^* - \left\lfloor \frac{1}{2} \sum_{i=1}^m s_i^{II} \right\rfloor\}, \bar{i} = i + \left\lceil \frac{L}{2} \right\rceil \text{ for } i = 0, \dots, l,$$

$b_i = p_{sII}^m(r^* - \bar{i})$ for $i = 0, \dots, l$ and

$$\left. \begin{aligned} A_i &= \frac{\bar{i} p_{sI}^n(\bar{i}) + (L - \bar{i}) p_{sI}^n(L - \bar{i})}{p_{sI}^n(\bar{i}) + p_{sI}^n(L - \bar{i})} \text{ for } i = 0, \dots, \bar{L}, \\ a_i &= p_{sI}^n(\bar{i}) + p_{sI}^n(L - \bar{i}) \text{ for } i = 0, \dots, \bar{L} \text{ if } L \text{ is odd or} \\ a_0 &= p_{sI}^n\left(\frac{L}{2}\right), a_i = p_{sI}^n(\bar{i}) + p_{sI}^n(L - \bar{i}) \text{ for } i = 1, \dots, \bar{L} \text{ if } L \text{ is even} \end{aligned} \right\} \text{ if } L >$$

0,

$$\left. \begin{aligned} A_i &= i + (r^* - l) \text{ and } a_i = p_{sI}^n(i + (r^* - l)) \\ &\text{for } i = 0, \dots, l \text{ if } L = 0 \text{ or} \\ &\text{for } i = \bar{L} + 1, \dots, l \text{ if } L > 0 \end{aligned} \right\} \text{ if } \bar{L} < l.$$

It is clear that $A_{\bar{L}} = \frac{L p_{sI}^n(L) + 0 p_{sI}^n(0)}{p_{sI}^n(L) + p_{sI}^n(0)}$ and

$$A_{\bar{L}+1} = \bar{L} + 1 + r^* - l = r^* - \left\lfloor \frac{1}{2} \sum_{i=1}^m s_i^{II} \right\rfloor + 1 + r^* - \left\lfloor \frac{1}{2} \sum_{i=1}^m s_i^{II} \right\rfloor = L + 1$$

if $L > 0$ ($\Rightarrow l = \left\lfloor \frac{1}{2} \sum_{i=1}^m s_i^{II} \right\rfloor$) and $\bar{L} < l$.

The symmetry $b_i = p_{sII}^m(r^* - \bar{i}) = p_{sII}^m\left(\sum_{i=1}^m s_i^{II} - r^* + \bar{i}\right) = p_{sII}^m(r^* - (L - \bar{i}))$

for $i = 0, \dots, \bar{L}$ if $L > 0$ (see (24)) leads to $\frac{\sum_{x=0}^{r^*} x p_{sI}^n(x) p_{sII}^m(r^* - x)}{\sum_{x=0}^{r^*} p_{sI}^n(x) p_{sII}^m(r^* - x)} =$

$$\frac{A_0 a_0 b_0 + \dots + A_l a_l b_l}{a_0 b_0 + \dots + a_l b_l}.$$

Using $r^* - \bar{i} \leq r^* - \left\lceil \frac{L}{2} \right\rceil \leq \left\lfloor \frac{1}{2} \sum_{i=1}^m s_i^{II} \right\rfloor$ for $i = 0, \dots, l$ together with

with (24) implies that $\{b_i\}$ is monotonically decreasing.

Obviously, $\{A_i\}$ is monotonically increasing if $L = 0$. In case $L > 0$ the

monotonicity follows from (28) since $p_{sI}^n(r^* - 1) \leq p_{sI}^n(r^*)$ implies

$$r^* \leq \max\{s_n, \left\lceil \frac{1}{2} \sum_{i=1}^n s_i \right\rceil\} \text{ (see (25)).}$$

We can now apply (29) and

$$\frac{\sum_{x=0}^{r^*} x p_{s_I}^n(x) p_{s_{II}}^m(r^*-x)}{\sum_{x=0}^{r^*} p_{s_I}^n(x) p_{s_{II}}^m(r^*-x)} = \frac{A_0 a_0 b_0 + \dots + A_l a_l b_l}{a_0 b_0 + \dots + a_l b_l} \leq \frac{A_0 a_0 + \dots + A_l a_l}{a_0 + \dots + a_l} = \frac{\sum_{x=0}^{r^*} x p_{s_I}^n(x)}{\sum_{x=0}^{r^*} p_{s_I}^n(x)}$$
 is proved.

To II: Firstly, $k = s_1 + \sum_{i=2}^{n+1} s_i \geq n+1$ follows from $|M_I| = m = 1$ and (14).

Then it is sufficient to show that $\frac{\sum_{x=0}^{r^*} x p_{s_I}^n(x)}{\sum_{x=0}^{r^*} p_{s_I}^n(x)} \leq r^* \frac{n}{n+1}$. This inequality can be proved by a simple mathematical induction using (27).

Proof of (7) in case $C[k, r^t]$:

In this case it must be showed that $\hat{f} = \max \{k, R(k) - k + 1\} =: C(k)$ satisfies the property (7).

We can use many ideas from case $C[r^t, k]$ in a similar way.

For this we replace $\left\{ \begin{smallmatrix} < \\ (-) \\ > \end{smallmatrix} \right\}$ by $\left\{ \begin{smallmatrix} > \\ (-) \\ < \end{smallmatrix} \right\}$ in (9), (14) and in the definitions of the sets M_I, \dots, M_{IV} and furthermore $\bullet - o$ by $o - \bullet$ in (9), (10), (12), (13), (15) and (22) (and in corresponding formulas without numbers) where $\bullet = 0$ is possible and the corresponding terms are shorter.

These manipulations implicate that r and s change roles in a way.

Moreover we substitute k by $C(k)$ in the conjectures.

In case $C[k, r^t]$ we must also keep $R(k)$ and $R(k) - k$ in mind.

The above mentioned manipulations lead to

$$Z_t(s, s') = \sum_{i: r_i^t < s_i} (s_i - r_i^t) \quad \forall s' \in \hat{A}_{N; k}(s, r^t).$$

The considered subsets of M are

$$\begin{aligned} M_I &= \{i \in M \mid s_i < s'_i \leq \bar{s}'_i = r_i^t \text{ or } s_i \leq s'_i < \bar{s}'_i = r_i^t\}, \\ M_{IIa} &= \{i \in M \mid s_i < s'_i \leq \bar{s}'_i < r_i^t \text{ or } s_i \leq s'_i < \bar{s}'_i < r_i^t\}, \\ M_{IIb} &= \{i \in M \mid s_i < \bar{s}'_i < s'_i \leq r_i^t\}, \\ M_{III} &= \{i \in M \mid r_i^t \geq s'_i > s_i \geq \bar{s}'_i \text{ or } r_i^t > s'_i = s_i \geq \bar{s}'_i\}, \\ M_{IV} &= \{i \in M \mid s_i \geq r_i^t = s'_i \geq \bar{s}'_i\} = \{i \in M \mid s_i \geq r_i^t\}. \end{aligned}$$

in case $C[k, r^t]$.

The first reduction steps also include the reduction of $R(k)$:

$R(k) = R(k) - \sum_{i \in M} \Delta_i$. (Note that the difference $R(k) - k$ remains the original one.)

Since $\Delta_i := \min \{s_i, r_i^t, \bar{s}'_i\}$ are others in case $C[k, r^t]$ the steps 2 and 3 are realized in the following way

2. $\bar{s} := \sum_{i \in M_{III}} (s_i - \bar{s}'_i) + \sum_{i \in M_{IV}} (r_i^t - \bar{s}'_i) = \sum_{i \in M_{III}} s_i + \sum_{i \in M_{IV}} r_i^t$,
 $s_i := 0 (= \bar{s}'_i), r_i^t := r_i^t - s_i, s'_i := s'_i - s_i$ for $i \in M_{III}$
 $s'_i = r_i^t := 0 (= \bar{s}'_i), s_i := s_i - r_i^t$ for $i \in M_{IV}$. Temporarily,
we set $s_{i'} = s'_{i'} = r_{i'}^t := \bar{s}, \bar{s}'_{i'} := 0$ for an additional $i' \in \{i'\} =: M_V$.
3. We replace the elements of M_{IV} by one element i where
 $s^* := \sum_{i \in M_{IV}} s_i, r_i^t := s'_i := \bar{s}'_i := 0$.

Then s^* takes the place of r^* .

An equation related to $R(k)$ must be added to (14):

$$k = s^* \leq \sum_{i \in M_I \cup M_{IIa} \cup M_{IIb}} \bar{s}'_i \leq \sum_{i \in M_I \cup M_{IIa} \cup M_{IIb} \cup M_{III}} r_i^t = R(k). \quad (32)$$

Furthermore $\sum_{i \in M_{III}} s_i \geq r^*$ (see (16)) is replaced by $\sum_{i \in M_{III}} r_i^t \geq s^*$.

$E_{s' \in \hat{A}_{N;k}(s,r)}(\hat{d}'(\bar{s}', \bar{s}')) \leq s^* - 1$ is sufficient (see (18)) since $s^* \leq k \leq C(k)$ according to (32).

Finally the reduced model with

$$\begin{aligned} 0 = s_i < s'_i \leq \bar{s}'_i = r_i^t \text{ or } 0 = s_i \leq s'_i < \bar{s}'_i = r_i^t \text{ for } i \in M_I, \\ r_i^t \geq s'_i > s_i = \bar{s}'_i = 0 \text{ or } r_i^t > s'_i = s_i = \bar{s}'_i = 0 \text{ for } i \in M_{III}, \\ s^* > r_i = s'_i = \bar{s}'_i = 0 \text{ for } i \in M_{IV}. \end{aligned}$$

is considered where the difference $R(k) - k$ calculated by the reduced $R(k)$ and k is less or equal to the original difference.

Analogous to (19) and (20)

$$k = s^*$$

and

$$s^* \leq \sum_{i \in M_I} \bar{s}'_i = \sum_{i \in M_I} r_i^t$$

are valid in case $C[k, r^t]$. In addition

$$\sum_{i \in M_{III}} r_i^t \leq R(k) - k. \quad (33)$$

A one-to-one correspondence between the elements of the sets $\hat{A}_{N;k}(s, r)$ and $P_{(r^{II}, r^I)}^{n+m}(s^*)$ is defined by:

$$\{s' \leftrightarrow \bar{x}\} \Leftrightarrow \{s'_i - s_i = s'_i = \bar{x}_i \text{ for } i = 1, \dots, n+m\},$$

where $r^I = (r_{m+1}^t, \dots, r_{m+n}^t), r^{II} = (r_1^t, \dots, r_m^t)$.

(23) is replaced by

$$E_{s' \in \hat{A}_{N;k}(s,r)}(\hat{d}'(s', \bar{s}')) = \frac{\sum_{x=0}^{s^*} x p_{rI}^n(x) p_{rII}^m(s^*-x)}{\sum_{x=0}^{s^*} p_{rI}^n(x) p_{rII}^m(s^*-x)} \leq s^* \frac{C(k)-1}{C(k)}.$$

Finally we use $C(k) \geq R(k) - k + 1 \geq |M_{III}| + |M_I| = n + 1$ which follows from (33) in order to prove II. \blacksquare

Remarks 1 *There is lot of theory about ordered restricted partitions of integers into **positive** parts (see [1] for example). In contrary for the case of ordered restricted partitions into **non-negative** parts not much results are known.*⁴ *Therefore we give some basic ideas for the proofs concerning the above used properties.*

- a) *The one-to-one correspondence $\bar{x} \leftrightarrow s - \bar{x}$ between the elements of the sets $P_s^n(x)$ and $P_s^n(\sum_{i=1}^n s_i - x)$ leads to the symmetry-property (24).*
- b) *The monotonicity (25) (and based on that the strict monotonicity (26)) can be proved by means of mathematical induction using (24) and the following simple recursive formula*

$$p_{(s_1, \dots, s_{n+1})}^{n+1}(x+1) = p_{(s_1, \dots, s_{n+1})}^{n+1}(x) + p_{(s_1, \dots, s_n)}^n(x+1) - p_{(s_1, \dots, s_n)}^n(x - s_{n+1}).$$

- c) *We now want to sketch the main idea for the proof of (27). Based on $P_s^n(x)$ the $r^* + 1$ sets $M_P^{r^*+1}(x)$ ($x = 0, \dots, r^*$) of partitions of $r^* + 1$ are generated in the following way:*

$M_P^{r^*+1}(x) := \{\bar{x} + (r^* + 1 - x)e_i \mid \bar{x} \in P_s^n(x), \bar{x}_i + (r^* + 1 - x) \leq s_i, i = 1, \dots, n\}$, where e_i is the i -th unit vector. (Each set $M_P^{r^*+1}(x)$ has at most $np_s^n(x)$ elements.) With that it remains to show that every partition of $P_s^n(r^* + 1)$ is thereby generated exactly $r^* + 1$ times.

- d) *Finally, we want to state the proof of (29):*

$$\begin{aligned} 0 &\leq \sum_{i=0}^l \sum_{j:j < i} a_i a_j (b_j - b_i) (A_i - A_j) \\ 0 &\leq \sum_{i=0}^l \sum_{j:j < i} [a_i a_j A_i (b_j - b_i) + a_j a_i A_j (b_i - b_j)] \\ \sum_{i=0}^l \sum_{j:j < i} (a_i a_j A_i b_i + a_j a_i A_j b_j) &\leq \sum_{i=0}^l \sum_{j:j < i} (a_i a_j A_i b_j + a_j a_i A_j b_i) \end{aligned}$$

⁴Although the method of the generating function and others can also be applied to ordered partitions into *non-negative* parts (s. Andrews).

$$\begin{aligned}
\sum_{i=0}^l \sum_{\substack{j=0 \\ j \neq i}}^l a_i a_j A_i b_i &\leq \sum_{i=0}^l \sum_{\substack{j=0 \\ j \neq i}}^l a_i a_j A_i b_j \\
\sum_{i=0}^l \sum_{j=0}^l a_i a_j A_i b_i &\leq \sum_{i=0}^l \sum_{j=0}^l a_i a_j A_i b_j \\
\frac{A_0 a_0 b_0 + \dots + A_l a_l b_l}{a_0 b_0 + \dots + a_l b_l} &\leq \frac{A_0 a_0 + \dots + A_l a_l}{a_0 + \dots + a_l}.
\end{aligned}$$

■

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